# All about the Chi-Squared Distibution

Sky Sjue

December 6, 2011

## 1 Motivation and derivation

This is a self-contained derivation of the $\chi^2$ probability distribution function (pdf) along with explanations of how to calculate its cumulative values. I find it distasteful being referred to this book or that book and constantly being given a result to take at face value: the *raison d'être* for this document. It should be completely accessible to anyone who knows calculus.

The Gaussian or normal pdf is ubiquitous. The Central Limit Theorem explains its frequent utility even when it is not the fundamental distribution of a given statistical problem. Consider the joint pdf of $N$ Gaussian variables.

$$dP = \prod_{i=1}^{N} \frac{dx_i}{\sqrt{2\pi\sigma_i^2}} \exp\left[-\frac{(x_i - \overline{x}_i)^2}{2\sigma_i^2}\right] \tag{1}$$

Substitute $y_i = (x_i - \overline{x}_i)/\sigma_i$.

$$dP = \prod_{i=1}^{N} \frac{dy_i}{\sqrt{2\pi}} \exp\left[-\frac{y_i^2}{2}\right] \tag{2}$$

This $N$-dimensional pdf in Cartesian coordinates may also be expressed in $N$-dimensional spherical coordinates.

$$dP = \frac{1}{(2\pi)^{N/2}} r^{N-1} dr d\Omega_{N-1} \exp\left[-\frac{r^2}{2}\right] \tag{3}$$

In this notation, $d\Omega_{N-1}$ represents an infinitesimal solid angle on the surface of a sphere in $N$ dimensions. Also, $r^2$ defines $\chi^2$:

$$r^2 = \sum_{i=1}^{N} \frac{(x_i - \overline{x}_i)^2}{\sigma_i^2} \equiv \chi^2 \tag{4}$$

The distribution with respect to $\chi^2$ is achieved by the substitution $t = \chi^2$.

$$\frac{dP}{dt} = \frac{\Omega_{N-1}}{2(2\pi)^{N/2}} t^{N/2-1} \exp(-t/2) \tag{5}$$

This concludes the derivation of the distribution. See Section 4 and Equation 29 for the value of $\Omega_{N-1}$. Inclusion of this result gives the concrete form of the $\chi^2$ pdf.

$$\frac{dP}{dt} = \frac{t^{N/2-1}}{2^{N/2}\Gamma(N/2)} \exp(-t/2) \tag{6}$$

If the factor $\Gamma(N/2)$ in the denominator is confusing, Section 5 gives the definition of the Gamma Function.

## 2 The mean, mode and asymptotic behavior of the $\chi^2$ pdf

Now let's calculate the expectation value of $\chi^2$.

$$\bar{t} = \int_0^\infty dtt\frac{dP}{dt} = \int_0^\infty dt\frac{t^{N/2}}{2^{N/2}\Gamma(N/2)} \exp(-t/2) \tag{7}$$

A canonical form for $\Gamma(z)$ follows from the substitution $z = t/2$.

$$\bar{t} = \int_0^\infty 2dz\frac{z^{N/2}}{\Gamma(N/2)} \exp(-z) = 2\frac{\Gamma(N/2+1)}{\Gamma(N/2)} \tag{8}$$

Integration by parts once on the definition for $\Gamma(z)$ given in Section 5 immediately shows $\Gamma(z+1) = z\Gamma(z)$. This gives the expectation value.

$$\bar{t} = \langle\chi^2\rangle = N \tag{9}$$

The mode of the $\chi^2$ pdf, where the probability is maximum, can be found by setting its derivative equal to zero. The functional dependence can be fully stuffed into the exponent.

$$\frac{dP}{dt} \propto \exp[f(t)] = \exp\left[(N/2-1)\ln t - t/2\right] \tag{10}$$

The derivative of the function in the exponent is $f' = (N/2-1)/t - 1/2$. Hence the most probable value is $\chi^2 = N - 2$.

The mode and mean of the distribution give a sort of bull's eye for test statistics. But the width and shape of the distribution around the mode varies. It is worth quickly mentioning the behavior of the distribution for large $N$. Taking another derivative of the function in the exponent to make a Taylor expansion, one finds

$$f'' = -\frac{N/2-1}{t^2} = -\frac{1}{2}\frac{N-2}{(N-2)^2} = -\frac{1}{2(N-2)} \tag{11}$$

This means that to leading order for large $N$, the distribution behaves as a Gaussian function centered at $N-2$ with $\sigma = \sqrt{2(N-2)} \approx \sqrt{2N}$.

# 3 Calculating the cumulative $\chi^2$ distribution

When using the value of $\chi^2$ to test a statistical hypothesis, it is useful to consider the cumulative distribution for $\chi^2$ given the number of degrees of freedom. An integral is necessary in this case.

$$P = \int_0^{\chi^2} dt \frac{t^{N/2-1}}{2^{N/2}\Gamma(N/2)} \exp(-t/2) \tag{12}$$

This integral takes a simple, familiar form (*well known by those who know it well*) if one substitutes $z = t/2$.

$$P = \int_0^{\chi^2/2} dz \frac{z^{N/2-1}}{\Gamma(N/2)} \exp(-z) \tag{13}$$

Aside from the gamma function, this expression for $P$ is a ringer for the partial gamma function:

$$\gamma(a, x) \equiv \int_0^x dz z^{a-1} \exp(-z) \tag{14}$$

All is well in the world of pdfs and cumulative pdfs, since $\gamma(a, x) \to \Gamma(a)$ as $x \to \infty$. So the cumulative $\chi^2$ distribution is given by the ratio of a partial gamma function to a gamma function.

$$P = \frac{\gamma(N/2, \chi^2/2)}{\Gamma(N/2)} \tag{15}$$

Now the problem is how to calculate the partial gamma function. Useful series expansions for $\gamma(a, x)$ are obtained from integration by parts. Using the formula $\int udv = [uv] - \int vdu$ and taking $\exp(-z)$ as $u$ yields the following result.

$$\gamma(a, x) = \int_0^x dz z^{a-1} \exp(-z) = \left[\frac{z^a}{a} \exp(-z)\right]_0^x - \int_0^x dz \frac{z^a}{a}[-\exp(-z)] \tag{16}$$

A pattern emerges with further integration to give a series expansion.

$$\gamma(a, x) = \exp(-x)\frac{x^a}{a}\left[1 + \frac{x}{a+1} + \frac{x^2}{(a+1)(a+2)} + \cdots\right] \tag{17}$$

This series converges quickly if $x$ is comparable with or smaller than $a$. For large values of $x$, it will still converge but slowly, since the terms in the series will be significant until the $(a + n)$ bit in the denominator grows larger than $x$ to make the contribution become small.

An expansion that converges quickly for large values of $x$ can be obtained by changing the limits on the integral and integrating by parts with $\exp(-z)dz$ taken to be $dv$.

$$\Gamma(a, x) = \int_x^\infty dz z^{a-1} \exp(-z) = \left[-z^{a-1} \exp(-z)\right]_x^\infty - \int_x^\infty dz (a-1) z^{a-2}[-\exp(-z)] \tag{18}$$

This expression uses $\Gamma(a,x)$ instead of $\gamma(a,x)$ because the limits of integration have changed. Note that $(\Gamma(a,x)+\gamma(a,x))/\Gamma(a) = 1$, so it is easy to get $\gamma(a,x)$ from $\Gamma(a,x)$. This series features $x$ in the denominator instead.

$$\Gamma(a,x) = \exp(-x)x^a \left[\frac{1}{x} + \frac{a-1}{x^2} + \frac{(a-1)(a-2)}{x^3} + \cdots\right] \tag{19}$$

This series is tricky, because if $a$ is not an integer then the numerator will eventually explode. But if $x \gg a$ it converges very quickly before the $a$ terms in the numerator become an issue.

These two series suffice to calculate the cumulative $\chi^2$ distribution for all values of $x$ and $a$. There is still a potential pitfall during computation because $\exp(-x)$ can be very small and $x^a$ can be very large. There is a simple solution, which is to start with the logarithms. Everything will be normalized by $\Gamma(a)$ in the end, so consider the following logarithm.

$$\ln\left[\frac{\exp(-x)x^a}{\Gamma(a)}\right] = -x + a\ln x - \ln(\Gamma(a)) \tag{20}$$

It is not at all inconvenient to do this, because most modern programming languages have math libraries which include a function that returns $\ln(\Gamma(z))$. There are routines that can be found to do this, too. The three terms together will yield a finite value when exponentiated.

# 4 The solid angle of an $N$-dimensional sphere

I first learned this trick from a homework problem in Professor Larry Yaffe's graduate E&M class at the University of Washington in 2001. Starting with a Gaussian, the goal is to compute this integral.

$$I = \int_{-\infty}^{\infty} dx \exp\left(-\frac{x^2}{2}\right) \tag{21}$$

One integral could never be enough, so let's make it two.

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy \exp\left(-\frac{x^2+y^2}{2}\right) \tag{22}$$

Now we transform to cylindrical coordinates.

$$I^2 = \int_0^{\infty} rdr \int_0^{2\pi} d\phi \exp\left(-\frac{r^2}{2}\right) = 2\pi \int_0^{\infty} rdr \exp\left(-\frac{r^2}{2}\right) \tag{23}$$

One substitution makes this integral a cinch. Trying $u = r^2$, one finds

$$I^2 = 2\pi \int_0^{\infty} \frac{du}{2} \exp(-u/2) = 2\pi \tag{24}$$

So that is an easy way to find the integral of a Gaussian (or rather, the normalization) $I = \sqrt{2\pi}$. There is no need to bother with a $\sigma$ because a very simple substitution takes care of it.

Now we boldly consider the integral of $N$ Gaussian distributions.

$$I^N = \prod_{i=1}^{N} \int_{-\infty}^{\infty} dx_i \exp\left(-\frac{x_i^2}{2}\right) = (2\pi)^{N/2} \tag{25}$$

This integral may also be expressed in $N$-dimensional spherical coordinates.

$$I^N = \int drr^{N-1} d\Omega_{N-1} \exp\left(-\frac{r^2}{2}\right) \tag{26}$$

The same $u = r^2$ substitution gives the integrand the form of a gamma function.

$$I^N = \Omega_{N-1} \int \frac{du}{2} u^{N/2-1} \exp(-u/2) \tag{27}$$

To make it plain as day for my simple mind, I put $t = u/2$ and find

$$I^N = \Omega_{N-1} 2^{N/2-1} \int dtt^{N/2-1} \exp(-t) = \Omega_{N-1} 2^{N/2-1} \Gamma(N/2) \tag{28}$$

Comparison of the two different methods gives the following solution for $\Omega_{N-1}$.

$$\Omega_{N-1} = \frac{2\pi^{N/2}}{\Gamma(N/2)} \tag{29}$$

# 5 The Gamma Function

The Gamma Function is a generalized version of the factorial function. One definition of it is:

$$\Gamma(z) = \int_0^{\infty} dtt^{z-1} \exp(-t) \tag{30}$$

Using integration by parts it is easy to show that if $z$ is an integer, $\Gamma(z) = (z-1)!$.

# 6 Books that annoyed me into writing this document

I will not list them and sufficient justification may be that paper is precious, but for some reason (a personality fault, presumably) my ratio of skepticism to faith is similar regardless of what I'm reading, whether it be a technical text or the Bible, the Koran, the Talmud, ... *whatever*.